Condensing and Extracting Against Online Adversaries

Eshan Chattopadhyay
Cornell University

Mohit Gurumukhani
Cornell University

Noam Ringach
Cornell University

Rocco Servedio
Columbia University

Part 0: Introduction

Randomness in computation

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- Useful for randomized algorithms, cryptography, distributed computing protocols, machine learning, etc.

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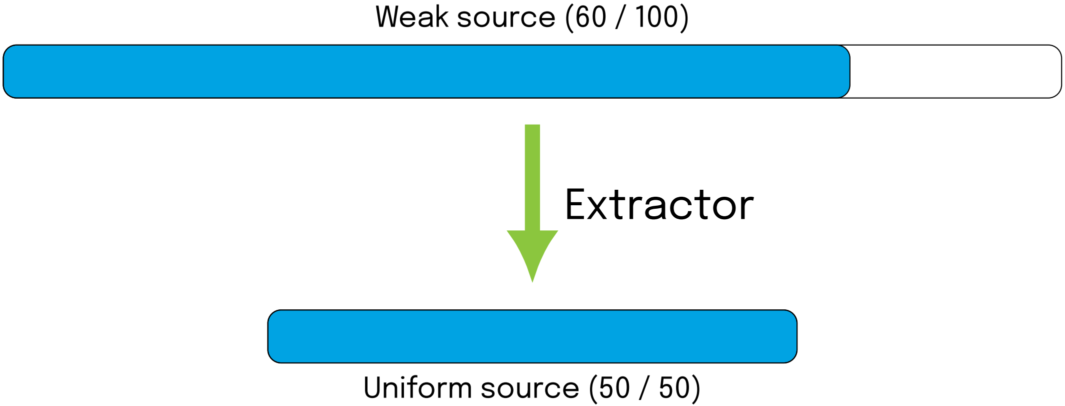
- Useful for randomized algorithms, cryptography, distributed computing protocols, machine learning, etc.
- Most applications need high quality randomness.
- In practice, randomness is derived from nature and is of low quality.

Extractor



Extractor

Definition (Extractor)

Ext : $\{0,1\}^n \to \{0,1\}^m$ is ε-extractor for class $\mathcal{X}$ if for all $\mathbf{X} \in \mathcal{X}$,

$$\left|\mathrm{Ext}(\mathbf{X}) - \textbf{Uniform}_m\right| \leq \varepsilon,$$

$|\cdot|$ denotes statistical distance / total variation distance:

$$|\mathbf{A} - \mathbf{B}| = \max_{S \subset \Omega} \left|\Pr\left(\mathbf{A} \in S\right) - \Pr\left(\mathbf{B} \in S\right)\right| = \frac{1}{2}\left\|\mathbf{A} - \mathbf{B}\right\|_1$$

Single extractor for every distribution?

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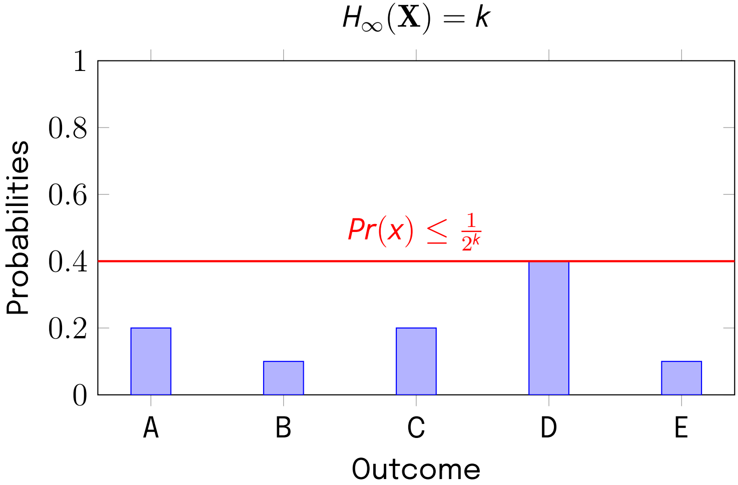
NO!

Single extractor for every distribution?

NO!

$\rightarrow$ **Distributions must have entropy**

Min-entropy


$H_\infty(\mathbf{X}) = k$
1
0.8
0.6
0.4
0.2
0
Probabilities
$Pr(x) \leq \frac{1}{2^k}$
A
B
C
D
E
Outcome

Min-entropy

Definition (Min-entropy)

Min-entropy of source $\mathbf{X}$:

$$H_\infty(\mathbf{X}) = -\log\left(\max_{x \in \mathsf{support}(\mathbf{X})} \Pr(\mathbf{X} = x)\right)$$

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Definition (Smooth Min-entropy)

Smooth min-entropy of source $\mathbf{X}$ with parameter ε:

$$H_\infty^\varepsilon(\mathbf{X}) = \max_{\mathbf{Y}:|\mathbf{X}-\mathbf{Y}|\leq\varepsilon} H_\infty(\mathbf{Y})$$

Single extractor for every high min-entropy distribution?

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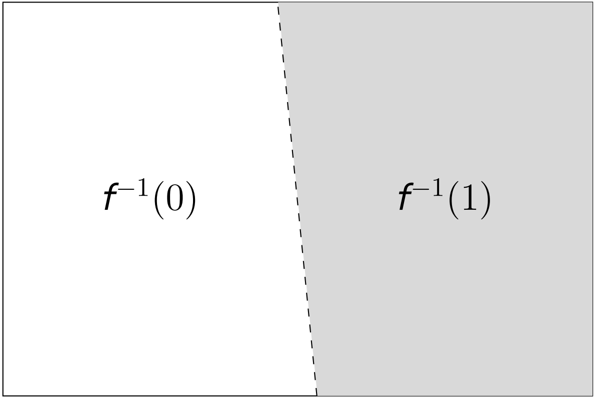
NO!

Single extractor for every high min-entropy distribution?



Single extractor for every high min-entropy distribution?

NO!

Solution: Distributions are structured.

Seedless extractors: a brief history

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- Two independent sources [Chor – Goldreich'88, ..., Li'23].
- Sources generated by circuits / low complexity classes (applications to circuit lower bounds) [Trevisan – Vadhan'00, ..., Viola'14, ...].
- Sources sampled by polynomials over large fields [Dvir – Gabizon – Wigderson'09, ...].
- Sources sampled by polynomials over $\mathbb{F}_2$ [Chattopadhyay – Goodman – Gurumukhani (CGG)'24].

Sometimes extractors don’t exist

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- Extractors guarantee closeness to uniform distribution.

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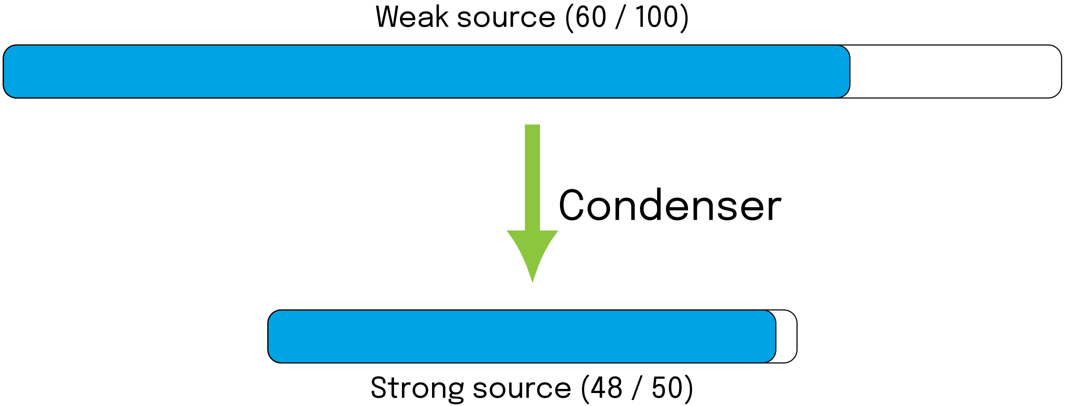
- Extractors guarantee closeness to uniform distribution.
- Relax this: guarantee closeness to high entropy distribution.

Condenser



Condenser

Condenser

Cond : $\{0,1\}^n \to \{0,1\}^m$ is a *$(k_{in}, k_{out}, \varepsilon)$-condenser* for class of distributions $\mathcal{X}$ with entropy at least k_{in} if for all $\mathbf{X} \in \mathcal{X}$,

$$H_\infty^\varepsilon(\text{Cond}(\mathbf{X})) \geq k_{out}$$

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- Care about increasing **entropy rate**:

$$\frac{k_{in}}{n} \text{ vs } \frac{k_{out}}{m}$$

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- Care about increasing **entropy rate**:

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- Care about minimizing **entropy gap**:

$$\Delta_{out} = m - k_{out}$$

Condensing is useful

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- Randomized algorithms with $\mathrm{poly}(\Delta_{out})$ overhead [Zuckerman'95].

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Simulating using only weak random source:

- Randomized algorithms with $\mathrm{poly}(\Delta_{out})$ overhead [Zuckerman'95].
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Condensers **can exist** where extractors (provably) **can't**

Single condenser for every high min-entropy distribution?

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NO!

Single condenser for every high min-entropy distribution?

NO!

→ **Same solution: Distributions should be structured.**

Seedless condensers: prior work

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- Condensers for Chor-Goldreich (CG) sources and adversarial Chor-Coldreich (CG) sources [Doron – Moshkovitz – Oh – Zuckerman'23].
- Improved Condensers for Chor-Goldreich Sources [Goodman – Li – Zuckerman'24]

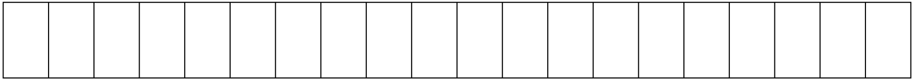
Part 1: Models and Results

OBFs

OBFs

Oblivious Bit Fixing Sources (OBFs)

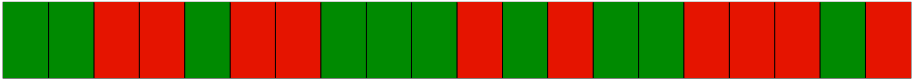
- ℓ bit input.



OBFs

Oblivious Bit Fixing Sources (OBFs)

- ℓ bit input.
- g good bits: uniform, $\ell - g$ bad bits: constants.



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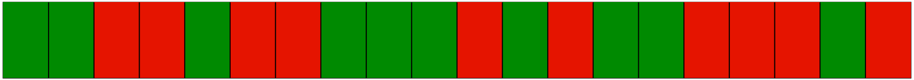


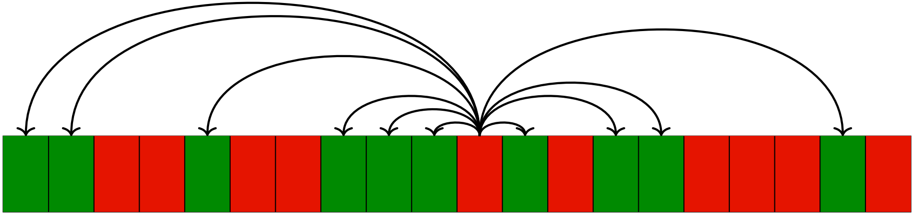
PARITY extracts from $(1, \ell)$-OBFs.

NOBFs

NOBFs

Non-Oblivious Bit Fixing Sources (NOBFs)

- g good bits: uniform, $\ell - g$ bad bits: arbitrary functions of good bits.



Extracting / Condensing from NOBFs

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Kahn – Kalai – Linial'88, Ben-Or – Linial'89, Ajtai – Linial'93

- **Can't** extract from $\left(\ell - \frac{\ell}{\log(\ell)}, \ell\right)$-NOBFs.

Extracting / Condensing from NOBFs

Kahn – Kalai – Linial'88, Ben-Or – Linial'89, Ajtai – Linial'93

- **Can't** extract from $\left(\ell - \frac{\ell}{\log(\ell)}, \ell\right)$-NOBFs.
- **Can** extract from $\left(\ell - \frac{\ell}{\log^2(\ell)}, \ell\right)$-NOBFs.

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Question

Can you condense from (g, ℓ)-NOBFs when $g < \ell - \frac{\ell}{\log \ell}$?

Extracting / Condensing from NOBFs

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- **Can** extract from $\left(\ell - \frac{\ell}{\log^2(\ell)}, \ell\right)$-NOBFs.

Theorem (Chattopadhyay – Gurumukhani – R (CGR)'24)

*For constant α, **can't** condense $((1-\alpha)\cdot\ell, \ell)$-NOBFs beyond **rate** $1-\alpha$.*

NOSFs

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Non-Oblivious Symbol Fixing Sources (NOSFs)

- ℓ blocks of length n each.

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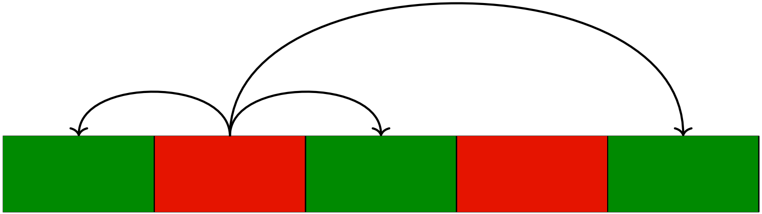
- ℓ blocks of length n each.
- g good blocks: uniform, $\ell - g$ bad blocks: arbitrary functions of good blocks.



Extracting / Condensing from NOSFs

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Aggarwal – Obremski – Ribeiro – Siniscalchi – Visconti (AORSV)’20

Can’t extract from $(0.99\ell, \ell)$-NOSFs.

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Extracting / Condensing from NOSFs

Aggarwal – Obremski – Ribeiro – Siniscalchi – Visconti (AORSV)'20

Can't extract from $(0.99\ell, \ell)$-NOSFs.

Theorem (CGR'24)

Can't *condense* (g, ℓ)*-NOSFs beyond* ***rate*** $\boldsymbol{g/\ell}$**.**

oNOSFs

oNOSFs

Online Non-Oblivious Symbol Fixing Sources (oNOSFs)

- g good blocks: uniform, $\ell - g$ bad blocks: arbitrary functions of good blocks **that appear before it**.

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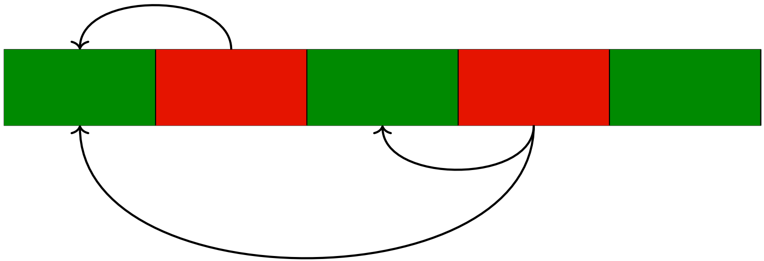

oNOSFs = Blockchain

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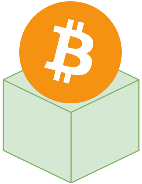

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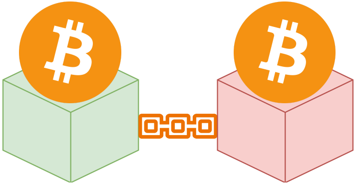

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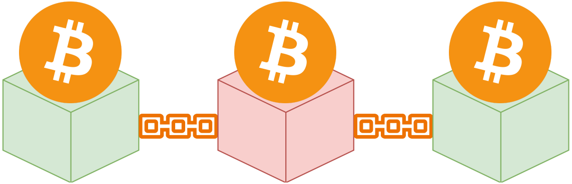

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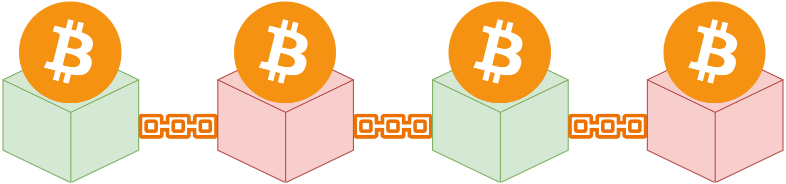

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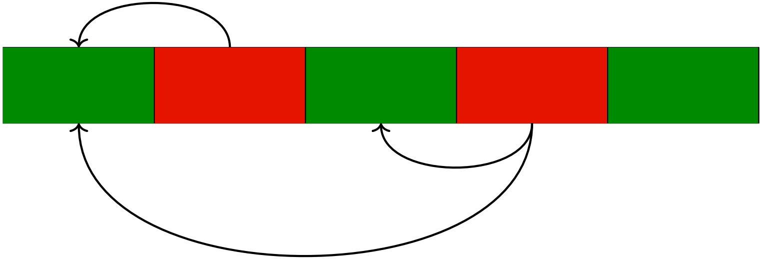

Extracting / Condensing from oNOSFs

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[AORSV'20]

Can't extract from $(0.99\ell, \ell)$-oNOSFs.

Extracting / Condensing from oNOSFs

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Question

Can you condense from (g, ℓ)-oNOSFs?

Extracting / Condensing from oNOSFs

[AORSV'20]

Can't extract from $(0.99\ell, \ell)$-oNOSFs.

Theorem (CGR'24, CGRS'25)

Can't *condense* (g, ℓ)*-oNOSFs beyond* ***rate*** $\boldsymbol{\frac{1}{\lfloor \ell / g \rfloor}}$**.**

Extracting / Condensing from oNOSFs

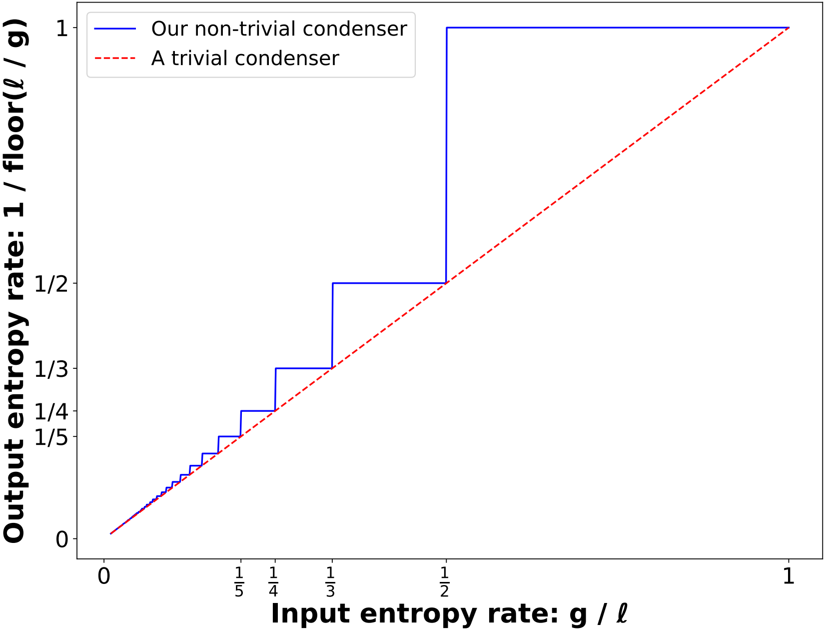
[AORSV'20]

Can't extract from $(0.99\ell, \ell)$-oNOSFs.

Theorem (CGR'24, CGRS'25)

Can't *condense* (g, ℓ)*-oNOSFs beyond* ***rate*** $\boldsymbol{\frac{1}{\lfloor \ell/g \rfloor}}$.

Can *condense* (g, ℓ)*-oNOSFs to* ***rate*** $\boldsymbol{\frac{1}{\lfloor \ell/g \rfloor}}$.


Our non-trivial condenser
A trivial condenser
Output entropy rate: 1 / floor(ℓ / g)
Input entropy rate: g / ℓ
1
1/2
1/3
1/4
1/5
0
0
$\frac{1}{5}$
$\frac{1}{4}$
$\frac{1}{3}$
$\frac{1}{2}$
1

Extracting / Condensing from oNOSFs

Corollary (Sharp threshold at $\boldsymbol{g} = \boldsymbol{\ell/2}$)

Can't *condense* $(0.5\ell, \ell)$*-oNOSFs beyond* ***rate*** $\mathbf{1/2}$ *- Impossibility.*
Can *condense* $(0.51\ell, \ell)$*-oNOSFs to* ***rate*** $\mathbf{0.99}$ *- Possibility.*

Part 2: Possibility

Condensers for (g, ℓ)-oNOSF sources

Theorem (Condensing uniform oNOSF sources)

For $g \geq 0.51\ell$, large constant block length n, and ℓ increasing, we can condense any oNOSF source to entropy rate 0.99.

Condensers for (g, ℓ)-oNOSF sources

Theorem (Condensing uniform oNOSF sources)

For $g \geq 0.51\ell$, $\ell = \Omega(\log(1/\varepsilon))$, *and* $n = 10^4$, *exists* $\mathsf{Cond} : \{0,1\}^{\ell n} \to \{0,1\}^m$ *s.t. for any* (g, ℓ)*-oNOSF* $\mathbf{X}$, $H_\infty^\varepsilon(\mathsf{Cond}(\mathbf{X})) \geq \mathbf{0.99}m$ *where* $m = \Omega(\ell + \log(1/\varepsilon))$.

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Theorem (Condensing low-entropy oNOSF sources)

For $g \geq 0.51\ell$, we can similarly condense oNOSF sources with logarithmic min-entropy.

Condensers for (g, ℓ)-oNOSF sources

Theorem (Condensing uniform oNOSF sources)

For $g \geq 0.51\ell$, $\ell = \Omega(\log(1/\varepsilon))$, *and* $n = 10^4$, *exists* Cond : $\{0,1\}^{\ell n} \to \{0,1\}^m$ *s.t. for any* (g, ℓ)*-oNOSF* $\mathbf{X}$, $\boldsymbol{H_\infty^\varepsilon}(\textbf{Cond}(\mathbf{X})) \geq \boldsymbol{0.99m}$ *where* $m = \Omega(\ell + \log(1/\varepsilon))$.

Theorem (Condensing low-entropy oNOSF sources)

For $g \geq 0.51\ell$, $n = \mathrm{polylog}(\ell/\varepsilon)$ *exists* Cond : $(\{0,1\}^n)^\ell \to \{0,1\}^m$ *s.t. for any* ***low-entropy*** (g, ℓ)*-oNOSF* $\mathbf{X}$ *with* $k = \Omega(\log(\ell/\varepsilon))$, $\boldsymbol{H_\infty^\varepsilon}(\textbf{Cond}(\mathbf{X})) \geq \boldsymbol{m - O(m/\log m) - O(\log(1/\varepsilon))}$ *where* $m = \Omega(k)$.

Condensers for (g,ℓ)-oNOSF sources

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Theorem (Extend AORSV'20 result)

Transform low-entropy (g,ℓ)*-oNOSFs* $\to$ *uniform* $(0.99g,\ell)$*-oNOSFs.*

Condensers for (g, ℓ)-oNOSF sources

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Does a random function work?

Condensers for (g, ℓ)-oNOSF sources

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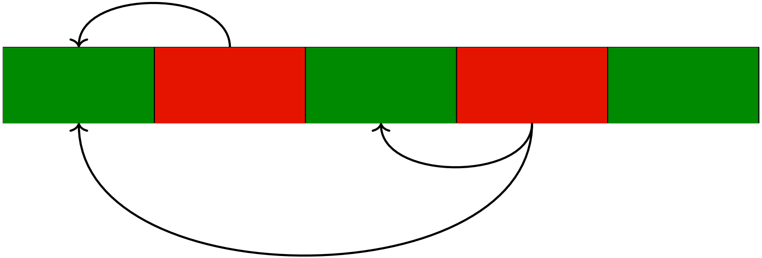
Does a random function work?

No!

Condensers for (g, ℓ)-oNOSF sources

Theorem (Condensing uniform oNOSF sources)

For $g \geq 0.51\ell$*,* $\ell = \Omega(\log(1/\varepsilon))$*, and* $n = 10^4$*, exists* Cond : $\{0,1\}^{\ell n} \to \{0,1\}^m$ *s.t. for any* (g, ℓ)*-oNOSF* $\mathbf{X}$*,* $\boldsymbol{H_\infty^\varepsilon}(\mathbf{Cond}(\mathbf{X})) \geq \mathbf{0.99}\boldsymbol{m}$ *where* $m = \Omega(\ell + \log(1/\varepsilon))$.



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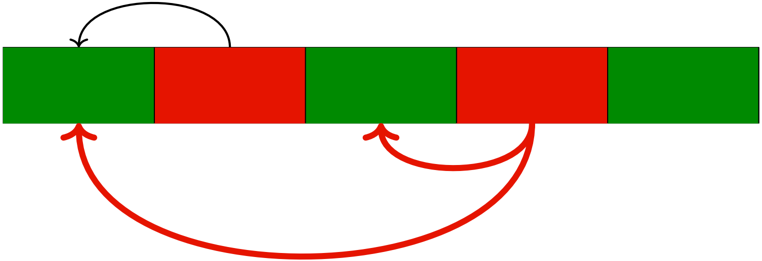

Constructing the condenser

Problem

A random function doesn't condense because the adversary has too much power in latter blocks.

$$\boxed{\mathbf{X}_1 \mid \quad \cdots \quad \mid \mathbf{X}_{\ell/2}} \qquad \boxed{\mathbf{X}_{\ell/2+1} \mid \quad \cdots \quad \mid \mathbf{X}_{\ell}}$$

Constructing the condenser

Solution

Take only first bit of latter half of blocks to weaken the adversary.

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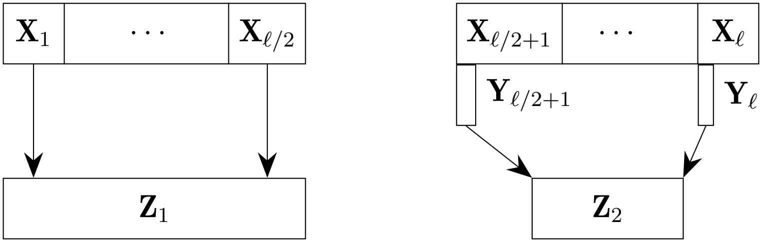

Constructing the condenser

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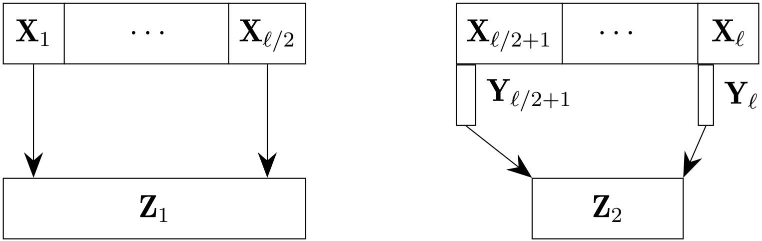
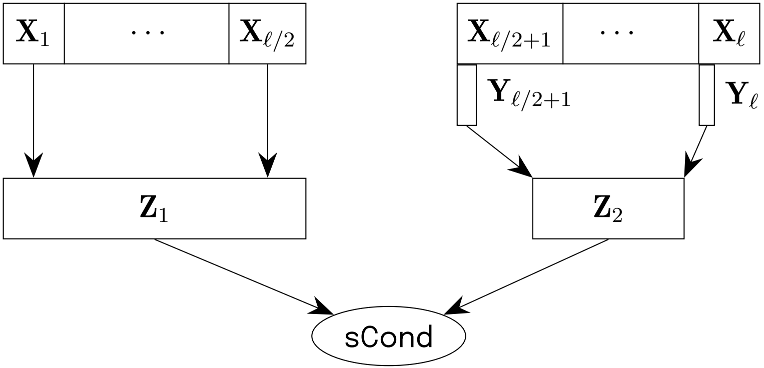


Now a random function works!

Constructing the condenser

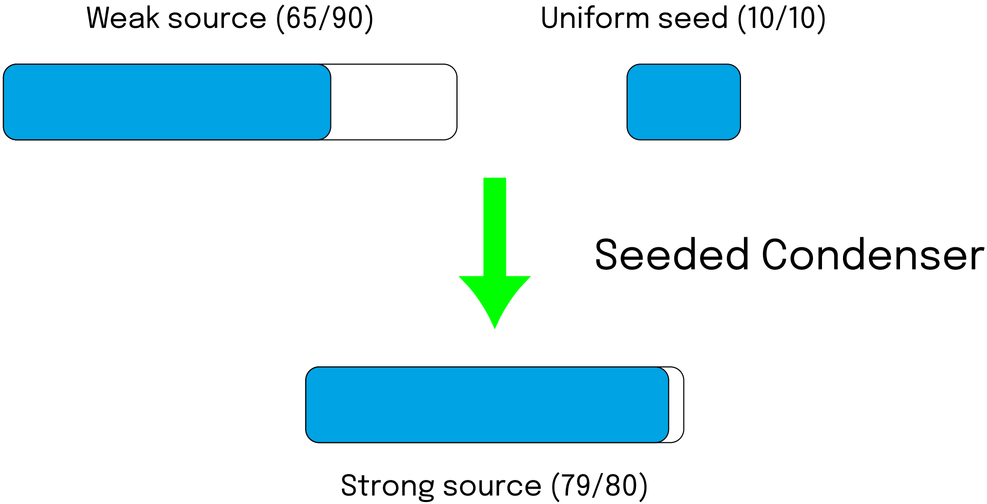
Solution

Take only first bit of latter half of blocks to weaken the adversary.



Seeded Condensers

Seeded Condensers



Seeded Condensers

Formal definition

A function sCond : $\{0,1\}^n \times \{0,1\}^d \to \{0,1\}^m$ is a *(k, ε)-seeded condenser* for a class of sources $\mathcal{X}$ if for all $\mathbf{X} \in \mathcal{X}$,

$$H_\infty^\varepsilon(\mathrm{sCond}(\mathbf{X}, \mathbf{U}_d)) \geq k$$

Seeded Condensers

Formal definition

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$$H_\infty^\varepsilon(\mathsf{sCond}(\mathbf{X}, \mathbf{U}_d)) \geq k$$

Theorem (Good seeded condensers exist)

Seeded condensers with logarithmic seed length and linear output length exist.

Seeded Condensers

Formal definition

A function $\mathsf{sCond} : \{0,1\}^n \times \{0,1\}^d \to \{0,1\}^m$ is a *(k, ε)-seeded condenser* for a class of sources $\mathcal{X}$ if for all $\mathbf{X} \in \mathcal{X}$,

$$H_\infty^\varepsilon(\mathsf{sCond}(\mathbf{X}, \mathbf{U}_d)) \geq k$$

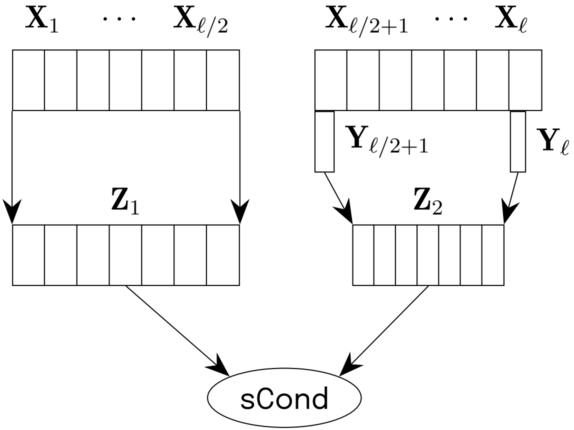
Theorem (Good seeded condensers exist)

For all d, ε s.t. $d \geq \log(\ell n/\varepsilon) + O(1)$ and $m = 0.01\ell n + d + \log(1/\varepsilon) + O(1)$, exists $\mathsf{sCond} : \{0,1\}^{\ell n/2} \times \{0,1\}^d \to \{0,1\}^m$ s.t. for all $\mathbf{X} \sim \{0,1\}^{\ell n/2}$ with $H_\infty(\mathbf{X}) \geq 0.01\ell n$, we have

$$H_\infty^\varepsilon(\mathsf{sCond}(\mathbf{X}, \mathbf{U}_d)) \geq 0.01\ell n + d$$

Correctness of the Condenser

Proof.



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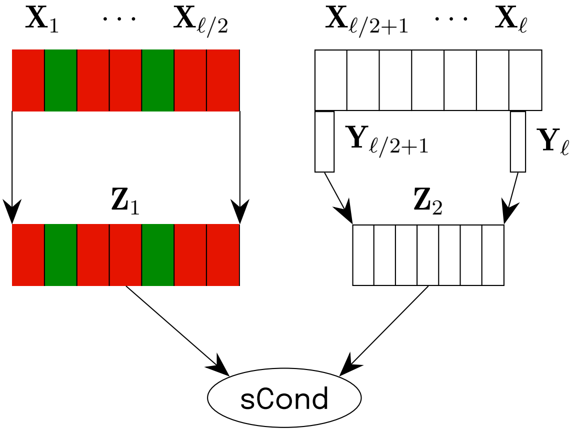

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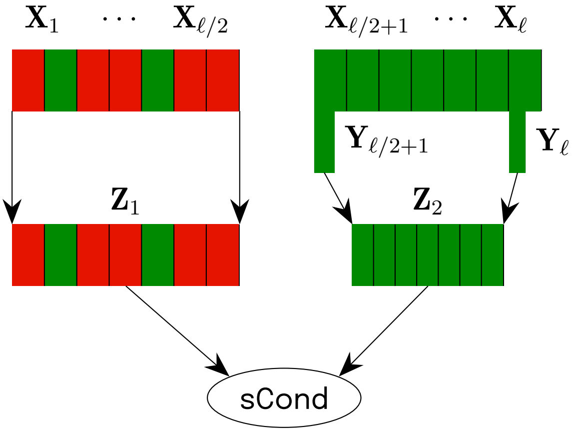

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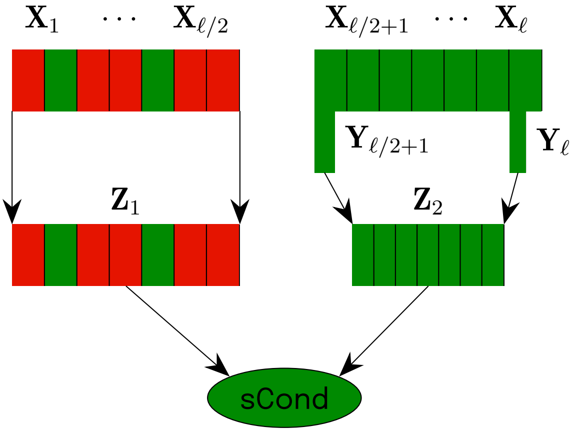

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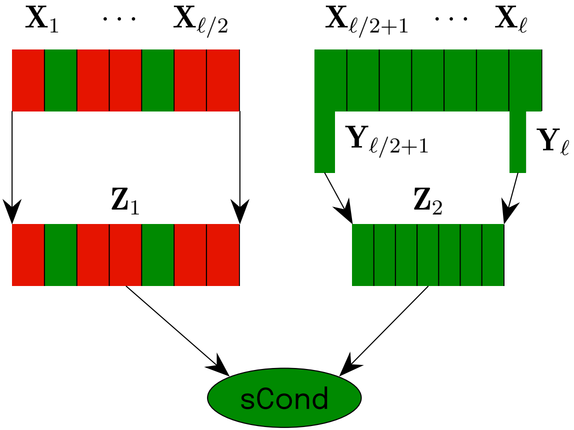

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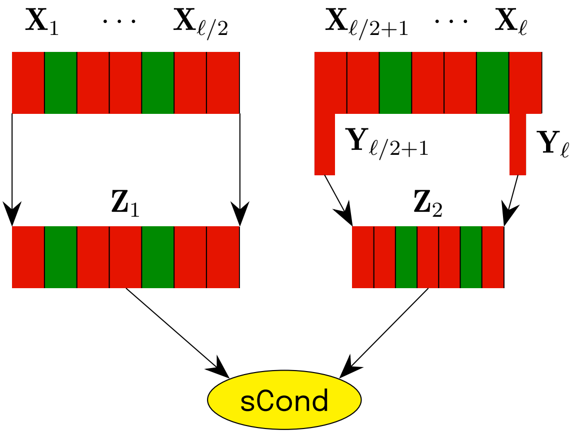

Adversary can’t make things too bad

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Lemma

Let $\mathbf{X} \sim \{0,1\}^n$ *and* sCond *be s.t.* $H_\infty^{\varepsilon_{\mathsf{sCond}}}(\mathsf{sCond}(\mathbf{X}, \mathbf{U}_d)) \geq k_{\mathsf{sCond}}$.
Let $\mathbf{U}'_d$ *be* $\mathbf{U}_d$ *except an adversary controls some b bits. Then,*

$$H_\infty^{\varepsilon'}(\mathsf{sCond}(\mathbf{X}, \mathbf{U}'_d)) \geq k_{\mathsf{sCond}} - b$$

where $\varepsilon' = \varepsilon_{\mathsf{sCond}} \cdot 2^b$.

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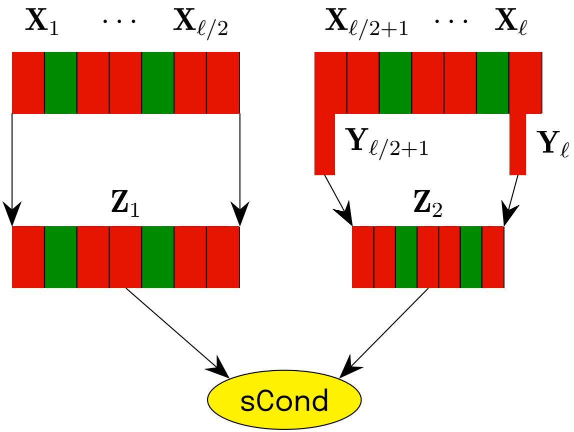

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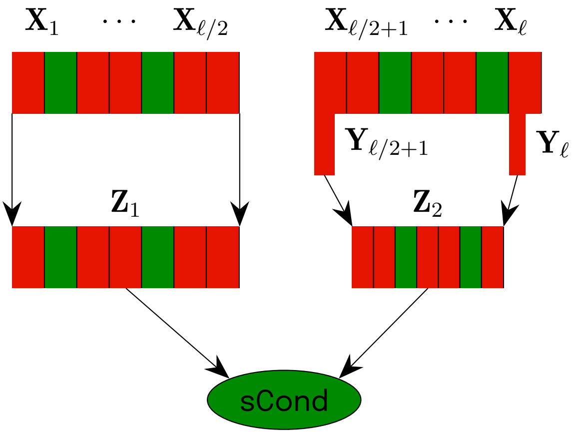

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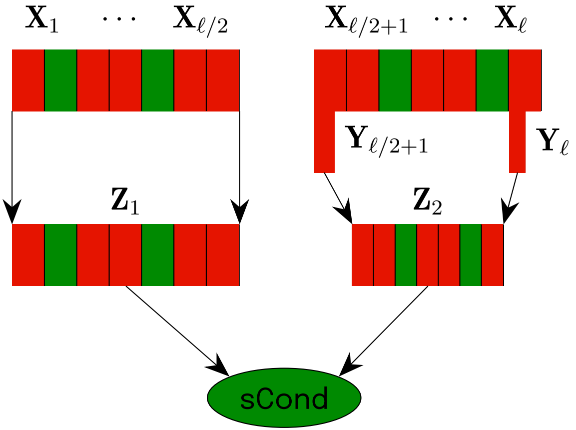

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- So for large const n, get entropy rate $\frac{m-O(\ell)}{m} \geq 0.99$.

Condensing from oNOSFs

Corollary (Sharp threshold at $\boldsymbol{g} = \boldsymbol{\ell/2}$)

Can't *condense* $(\ell/2, \ell)$*-oNOSFs beyond* ***rate*** $\mathbf{1/2}$ *- Impossibility.*
Can *condense* $(0.51\ell, \ell)$*-oNOSFs to* ***rate*** $\mathbf{0.99}$ *- Possibility.* $\checkmark$

Part 3: Future Directions

Open questions

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